

On the Query Complexity of Real Functionals

Hugo Férée, Walid Gomaa, Mathieu Hoyrup

- 1 Introduction
- 2 Complexity of Norms
- 3 Query Complexity
- 4 One Oracle Access

- 1 Introduction
- 2 Complexity of Norms
- 3 Query Complexity
- 4 One Oracle Access

Computing with Real Numbers

$$(q_n)_{n \in \mathbb{N}} \rightsquigarrow x \text{ if } \forall n, |x - q_n| \leq 2^{-n} \quad (1)$$
$$\mathbb{R} \sim (\mathbb{N} \rightarrow \mathbb{N})$$

Model: **Oracle Turing Machines**

Finite-time computation \longrightarrow finite number of queries
 \longrightarrow finite knowledge of the input
 \longrightarrow continuity

Bounding computation time \iff bounding

- the computational power
- the number of queries

Computing with Real Numbers

$$(q_n)_{n \in \mathbb{N}} \rightsquigarrow x \text{ if } \forall n, |x - q_n| \leq 2^{-n} \quad (1)$$

$$\mathbb{R} \sim (\mathbb{N} \rightarrow \mathbb{N})$$

Model: **Oracle Turing Machines**

Finite-time computation \longrightarrow finite number of queries
 \longrightarrow finite knowledge of the input
 \longrightarrow continuity

Bounding computation time \iff bounding

- ~~the computational power~~
- **the number of queries**

Computing with Real Numbers Cont'd

$$f \in \mathcal{C}[0, 1] \iff \exists \mu, f_{\mathbb{Q}} :$$

modulus of continuity : $\mu : \mathbb{N} \rightarrow \mathbb{N}$

$$|x - y| \leq 2^{-\mu(n)} \implies |f(x) - f(y)| \leq 2^{-n}$$

approximation function $f_{\mathbb{Q}} : \mathbb{Q} \times \mathbb{N} \rightarrow \mathbb{Q}$

$$|f_{\mathbb{Q}}(q, n) - f(q)| \leq 2^{-n}$$

Computing with Real Numbers Cont'd

Theorem

$f : \mathbb{R} \rightarrow \mathbb{R}$ is computable w.r.t. an oracle
 $\iff f$ is continuous.

Theorem

$f : \mathbb{R} \rightarrow \mathbb{R}$ is polynomial time computable w.r.t. to an oracle \iff
its modulus of continuity is bounded by a polynomial.

- 1 Introduction
- 2 Complexity of Norms
- 3 Query Complexity
- 4 One Oracle Access

Dependence of a Norm on a Point

F is a norm over $\mathcal{C}[0, 1]$

$f \in \mathcal{C}[0, 1], \alpha \in [0, 1]$

$\Delta\alpha$ implies $\Delta F(f)$

Dependence of a Norm on a Point

Two problems

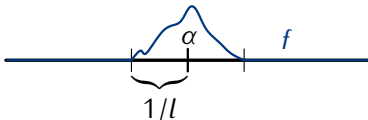
- 1 f is continuous, so must change a neighborhood of α ,
- 2 if $f = 0$, any change causes $F(f) = 0$ to change to some positive value.

Dependence of a Norm on a Point

Quantitative measure

F is always assumed weaker than the uniform norm

$$d_{F,\alpha}(n) = \sup\{l : \exists f \in Lip_1, Supp(f) \subseteq \mathcal{N}(\alpha, 1/l) \text{ and } F(f) > 2^{-n}\}$$

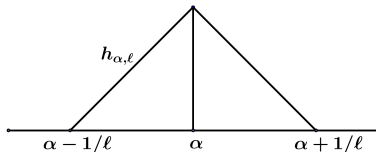


$$d_{F,\alpha} \leq c2^n, \text{ non-decreasing, unbounded}$$

Examples

The uniform norm

The uniform norm is monotonic:

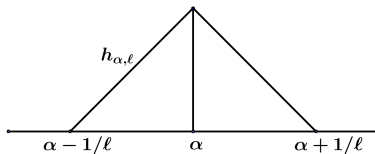


$$F = \|\cdot\|_{\infty} \implies d_{F, \alpha}(n) \sim 2^n \quad (2)$$

Examples

The L_1 -norm

- The L_1 -norm is monotonic:



$$F = \|\cdot\|_1 \implies d_{F, \alpha}(n) \sim 2^{\frac{n}{2}} \quad (3)$$

Some Properties of the Dependency Function

Proposition

- 1 $\alpha \mapsto d_{F,\alpha}(n)$ is continuous
- 2 F is weaker than $G \implies d_{F,\alpha}(n) \leq d_{G,\alpha}(n+k)$

Maximal dependence: $D_F(n) = \max_{\alpha \in [0,1]} d_{F,\alpha}(n)$

Proposition

For F weaker than the uniform norm:

$$c_1 2^{\frac{n}{2}} \leq D_F(n) \leq c_2 2^n \quad (4)$$

Relevant Points

$$R_{n,l} = \{\alpha : d_{F,\alpha}(n) \geq l\}$$

Definition

α is **relevant** if $\exists c > 0, \forall n, \quad d_{F,\alpha}(n) \geq c \cdot 2^{\frac{n}{2}}$

$$\mathcal{R} = \bigcup_k \underbrace{\bigcap_n R_{n, 2^{\frac{n}{2}-k}}}_{\mathcal{R}_k}$$

Example

For $\|\cdot\|_\infty$ and $\|\cdot\|_1$, $\mathcal{R} = [0, 1]$.

Relevant Points

Example

- Let $Q = \{q_0, q_1, \dots\}$ be some particular canonical enumeration of the dyadic rationals
- Define

$$F(f) = \sum_i 2^{-i} |f(q_i)|$$

Then

$$d_{F, q_i}(n) \geq 2^{n-i}, \quad \text{for } n \geq i$$

$$d_{F, \alpha}(n) \leq \frac{n^2}{\epsilon}$$

$$\mathcal{R} = \mathbb{D}$$

Relevant Points

Properties

Theorem

\mathcal{R} is dense.

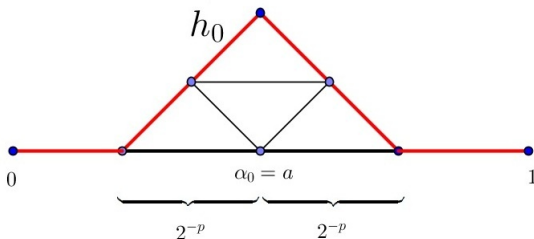
Theorem

$f = 0$ on $\mathcal{R}_{2\mu_f(k)} \implies F(f) \leq c \cdot 2^{-k}$.

Corollary

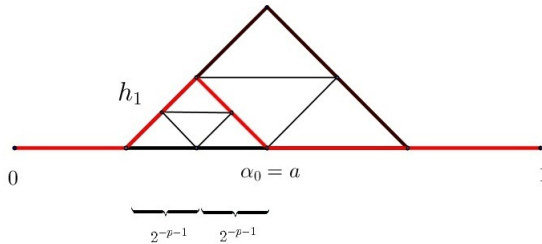
$f = g$ on $\mathcal{R}_{\mu(k)} \implies |F(f) - F(g)| \leq 2^{-k}$.

\mathcal{R} is Dense (Proof)



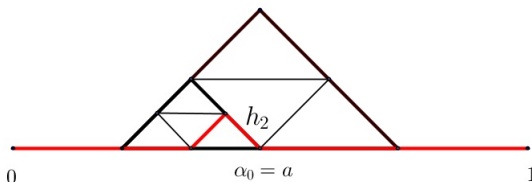
$$F(h_0) \geq 2^{-c}$$

\mathcal{R} is Dense (Proof)



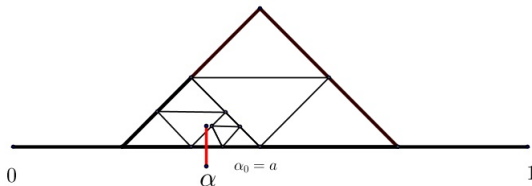
$$F(h_1) \geq 2^{-c-2}$$

\mathcal{R} is Dense (Proof)



$$F(h_2) \geq 2^{-c-4}$$

\mathcal{R} is Dense (Proof)



$$F(h_n) = F(h_{\alpha_n, 2^{-n-p}}) \geq 2^{-c-2n} \implies F(2n+c) \geq 2^{n+p}$$

$$(\alpha_n) \rightarrow \alpha$$

$$d_{F,\alpha}(2n+c) \geq 2^{n+p-1} \implies \alpha \in \mathcal{R}$$

- 1 Introduction
- 2 Complexity of Norms
- 3 Query Complexity**
- 4 One Oracle Access

Query Complexity

Definition

Q_n : oracle calls of F on $x \mapsto 0$ with precision 2^{-n} .

Proposition

$$R_{n,l} \subseteq \mathcal{N}(Q_{n+1}, \frac{1}{l})$$

Definition

F has a polynomial query complexity if it is computable by a relativized OTM with $|Q_n| \leq P(n)$.

Query Complexity Cont'd

Theorem

If F has polynomial query complexity, then almost every point has a polynomial dependency ($d_{F,\alpha} \in \mathcal{P}$ for almost all α).

Theorem

If F has polynomial query complexity, then \mathcal{R} has Hausdorff dimension 0.

Proposition

F has polynomial query complexity
 $\implies \exists \alpha, \frac{2^n}{d_{F,\alpha}(n)}$ is bounded by a polynomial.

Query Complexity

Characterizing polynomial time computable norms

Theorem

F is polynomial time computable w.r.t. an oracle

\iff *F has polynomial query complexity*

\iff *\mathcal{R}_k can be polynomially covered (wrt. k).*

$$\mathcal{R}_k = \{\alpha \in [0, 1]: \forall n, d_{F,\alpha}(n) \geq 2^{\frac{n}{2}-k}\} = \bigcap_n R_{n, 2^{\frac{n}{2}-k}}$$

Open question

Can it be generalized for any $F : \mathcal{C}[0, 1] \rightarrow \mathbb{R}$?

- 1 Introduction
- 2 Complexity of Norms
- 3 Query Complexity
- 4 One Oracle Access

One Oracle Access Case

Theorem

The following are equivalent:

- F is computable by a polynomial time machine doing only one oracle query
- $\forall f, F(f) = \phi(f(\alpha))$ where:
 - $\alpha \in \text{Poly}(\mathbb{R})$ (but cannot be efficiently retrived from F !)
 - $\phi \in \text{Poly}(\mathbb{R} \rightarrow \mathbb{R})$
 - ϕ is uniformly continuous

Open question

Generalization to any finite number of queries?

THANK YOU